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OPTIMAL PLANNING OF MEASUREMENTS IN NUMERICAL EXPERIMENT DETERMINATION OF THE CHARACTERISTICS OF A HEAT FLUX

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The authors present an algorithm and analyze results of optimization of a temperature measurement scheme for solving inverse heat-conduction boundary problems.

In experimental investigations and the development of thermal regimes for various thermally loaded engineering items, there has recently been wide use of methods of diagnosing heat fluxes based on solving inverse heat-conduction boundary problems (IBP) [1]. The use of these methods requires careful analysis of the computing properties of the IBP solution algorithm (e.g., rate of convergence, stability, errors in recovering the desired functions) and determining the conditions for conducting the temperature measurements to achieve maximum reliability of results of the diagnosis.

The mathematical modeling data show that the accuracy of recovering the boundary thermal conditions can be increased by choosing the location of the thermal sensors in the test body, and also by solving the IBP in a redefined formulation [2]. Here the question arises of the baseline choice of the number of thermal sensors and their rational location in the specimen. The present paper analyzes this problem from the standpoint of theory of an optimal experiment [3, 4].

We consider a planar unbounded plate of thickness b in which the heat-transfer process is described by the following equation of unsteady heat conduction with boundary conditions of the second kind:

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ 0<x<b,\ 0<\tau\leqslant\tau_m, \tag{1}$$

$$T(x,\ 0)=T_0(x),\ 0\leqslant x\leqslant b, \tag{2}$$

$$-\lambda(T(0,\ \tau))\frac{\partial T(0,\ \tau)}{\partial x}=q_1(\tau), \tag{3}$$

$$-\lambda(T(b,\ \tau))\frac{\partial T(b,\ \tau)}{\partial x}=q_2(\tau). \tag{4}$$

The IBP consists of defining the heat-flux density on one of the boundaries, e.g., $q_1(\tau)$, or simultaneously on both boundaries, $q_1(\tau)$ and $q_2(\tau)$, using the mathematical model of Eqs. (1)-(4) and the measured temperature data at a certain limited number N of points of the plate with coordinates $x=X_i,\ i=\overline{1,\ N}$:

$$T^{\exp}(X_i,\ \tau)=f_i(\tau),\ i=\overline{1,\ N}. \tag{5}$$

Efficient iterative computing algorithms for recovering the above characteristics have been proposed, for example, in [1, 2], in which the approximate solution of the inverse problem is determined from the uncoupling condition:

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$$I = \sum_{i=1}^{N} \int_{0}^{\tau_m} [T(X_i, \tau) - f_i(\tau)]^2 d\tau \simeq \delta^2, \quad (6)$$

where $T(x, \tau)$ is the temperature calculated with the aid of the mathematical model of Eqs. (1)-(4) for fixed heat loads; $\delta^2 = \sum_{i=1}^{N} \int_{0}^{\tau_m} \sigma^2(X_i, \tau) d\tau$ is the generalized error of the temperature measurements; and $\sigma^2(X_i, \tau)$ is the variance of the error of measuring temperature at the point with coordinate $x = X_i$.

From the conditions of uniqueness of the solution of the IBP to recover one function it is sufficient to take an unsteady temperature measurement at one spatial point. To recover two characteristics, the minimum necessary experimental information is temperature measurement at two points [1]. A greater number of thermal sensors leads to an overdefined formulation of the IBP.

We introduce parameterization of the unknown characteristics in the form

$$q_r(\tau) = \sum_{j=1}^{m_r} p_{rj} \varphi_{rj}(\tau), \quad (7)$$

where $q_r(\tau)$ is the desired characteristic; $r = 1, 2$, number of the boundary on which the characteristic is being recovered; and $\varphi_{rj}(\tau),\ j = \overline{1, m_r}$, system of baseline functions. Then the inverse problem reduces to determining (evaluating) the parameter vector $\mathbf{P} = \{p_k\}_1^M$, $M = m_1 + m_2$, which includes coefficients of the parameteric representation of the recovered (one or two) functions. If the heat flux density is not recovered on the r-th boundary, then $m_r \equiv 0$.

We shall now formulate the problem of optimal planning of the measurements. For this purpose we introduce the measurement plan or scheme

$$\varepsilon = \{N, \mathbf{X}\}, \ \mathbf{X} = \{X_i\}_1^N. \quad (8)$$

A rational choice of the measurement scheme is to use some scalar quality index describing the accuracy of recovering the unknown parameter vector **P**. In analysis of an optimal experiment for inverse problems of mathematical physics a quality index of this type that finds wide use is the D-optimum criterion [4]

$$\Psi = \det [F(\varepsilon)], \quad (9)$$

where $F(\varepsilon)$ is the normalized Fisher information matrix:

$$F(\varepsilon) = \frac{1}{N} \{\Phi_{kj};\ k, j = \overline{1, M}\}; \quad (10)$$

$$\Phi_{kj} = \sum_{i=1}^{N} \int_{0}^{\tau_m} \sigma^2(X_i, \tau) \Theta_k(X_i, \tau) \Theta_j(X_i, \tau) d\tau;$$

$\Theta_k(x, \tau) \equiv \dfrac{\partial T(x, \tau)}{\partial p_k}$, $k = \overline{1, M}$ is the sensitivity function.

The matrix $F(\varepsilon)$ characterizes the total sensitivity of the system at the measurement points X_i, $i = \overline{1, N}$, to small variations of the entire set of parameters p_k, $k = \overline{1, M}$. We require to find a measurement plan ε^* for which the criterion (9) reaches a maximum value. Here the region of allowable values of the sensor location coordinates is determined by the geometric size of the test specimen. A similar problem also arises in optimizing measurements for the coefficients of inverse heat-conduction problems [5, 6].

Thus, a rational choice of the measurement scheme leads to the need to solve the following extremal problem:

$$\varepsilon^* = \operatorname{Arg} \max \det F(\varepsilon), \ \ \varepsilon = \{N, \mathbf{X}\}, \ \mathbf{X} = \{X_i\}_1^N, \ 0 \leqslant X_i \leqslant b, \ i = \overline{1, N}. \quad (11)$$

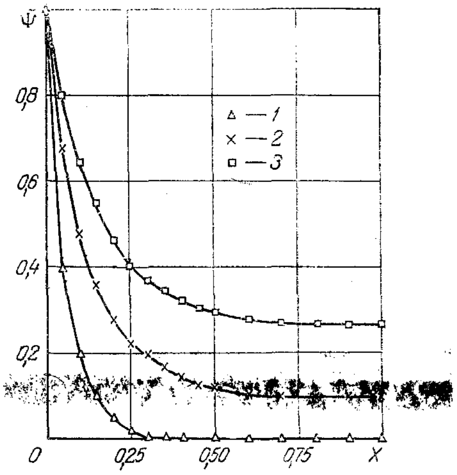

Fig. 1

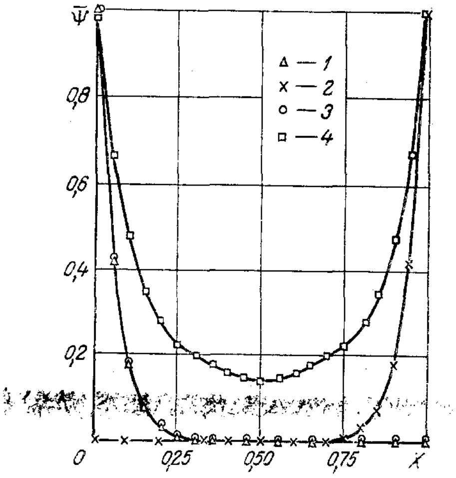

Fig. 2

Fig. 1. Parameter $\bar{\Psi}$ as a function of thermal sensor location for the recovery of heat-flux density $q_1(\tau): 1 - N=1,\ 0\leqslant X_1\leqslant 1;\ 2 - N=2,\ X_1^*=0,\ 0\leqslant X_2\leqslant 1;\ 3 - N=3,\ X_1^*=X_2^*=0,\ 0\leqslant X_3\leqslant 1$ ($\Psi_{max}=0,\ 1023\cdot 10^{-11}$)

Fig. 2. Parameter $\bar{\Psi}$ as a function of thermal sensor location with simultaneous recovery of heat flux densities $q_1(\tau)$ and $q_2(\tau)$. For N = 2: 1) $1 - 0\leqslant X_1\leqslant 1,\ X_2=1;\ 2 - X_1^*=0,\ 0\leqslant X_2\leqslant 1$. For $N=3$: $3 - 0\leqslant X_1\leqslant 1,\ X_2^*=X_3^*=1;\ 4 - X_1^*=0,\ 0\leqslant X_2\leqslant 1,\ X_3^*=1$ ($\Psi_{max}=0,3887\cdot 10^{-26}$)

The solution of this problem can be constructed by carrying out an iterative procedure in which at each iteration one solves the problem of finding an optimal vector of the coordinates of a fixed number of sensors N:

$$\mathbf{X}^* = \operatorname{Arg\,max} \det F(N,\ \mathbf{X}),\ 0\leqslant X_i\leqslant b,\ \ i=\overline{1,\ N}, \tag{12}$$

and successively increases the number of sensors by one. The iterative process ends when one satisfies the condition

$$|(\Psi(l+1,\ \mathbf{X}^*)-\Psi(l,\ \mathbf{X}^*))/\Psi(l+1,\ \mathbf{X}^*)|\leqslant\rho,$$

where l is the iteration number; and $\rho > 0$ is the given quantity. Here the minimum number of sensors is determined from the condition that a unique solution exists for the IBP being analyzed. The result is that one constructs a measurement scheme which ensures the maximum accuracy of recovering the boundary heat fluxes.

We now consider special features of the computing algorithm for solving the extremal problem of Eq. (12). To determine the elements of the information matrix (10), we need to calculate the sensitivity function $\Theta_k(x, \tau)$, $k=\overline{1, M}$. For this purpose we solve M boundary problems which are obtained by differentiating the original problem of Eqs. (1)-(4) with respect to all the parameters p_k, $k = \overline{1, M}$. In the case considered, the boundary problems for the sensitivity function have the form:

$$C(T)\frac{\partial\Theta_k}{\partial\tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial\Theta_k}{\partial x}\right)+\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial x}\frac{\partial\Theta_k}{\partial x}+\left[\frac{\partial^2 T}{\partial x^2}\frac{\partial\lambda}{\partial T}+\right.$$
$$\left.+\left(\frac{\partial T}{\partial x}\right)^2\frac{\partial^2\lambda}{\partial T^2}-\frac{\partial T}{\partial\tau}\frac{\partial C}{\partial T}\right]\Theta_k,\ 0<x<b,\ 0<\tau\leqslant\tau_m,\ k=\overline{1,\ m_r},\ r=1,\ 2, \tag{13}$$

$$\Theta_k(x,\ 0)=0,\ \ 0\leqslant x\leqslant b, \tag{14}$$

$$\lambda(T(0,\ \tau))\frac{\partial\Theta_k(0,\ \tau)}{\partial x}+\frac{\partial T(0,\ \tau)}{\partial x}\frac{\partial\lambda(T(0,\ \tau))}{\partial T}\Theta_k(0,\ \tau)=-\delta_{r1}\varphi_{1k}(\tau), \tag{15}$$

$$\lambda(T(b,\ \tau))\frac{\partial\Theta_k(b,\ \tau)}{\partial x}+\frac{\partial T(b,\ \tau)}{\partial x}\frac{\partial\lambda(T(b,\ \tau))}{\partial T}\Theta_k(b,\ \tau)=-\delta_{r2}\varphi_{2,k-m_1}(\tau), \tag{16}$$

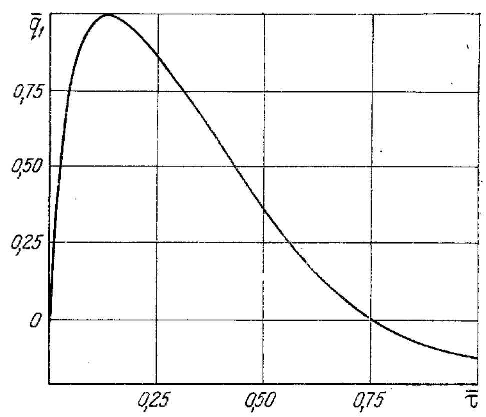

Fig. 3

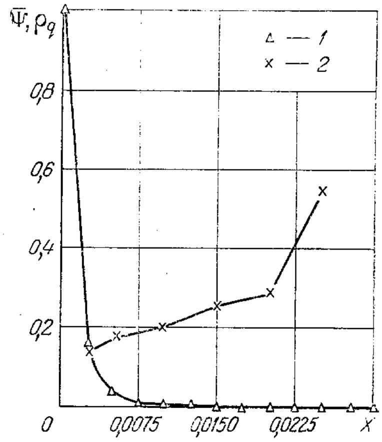

Fig. 4

Fig. 3. Variation of heat flux density at the plate boundary.

Fig. 4. Relative value of the planning criterion (1) and the relative integral error (2) in solving the inverse boundary problem, as a function of thermal sensor location ($\Psi_{max}=0{,}1958\cdot 10^{-5}$).

where δ_{rj} is the Kronecker delta; and $\delta_{rj}=\begin{cases}1 & \text{for } r=j,\\ 0 & \text{for } r\neq j.\end{cases}$

It should be noted that the sensitivity function, and therefore the optimal measurement plan ε^* depend on the unknown parameter vector **P**. The reason is that the temperature field $T(x, \tau)$, $0\leqslant x\leqslant b$, $0\leqslant\tau\leqslant\tau_m$, which is determined by solving the boundary problem of Eqs. (1)-(4), depends nonlinearly on the unknown parameters. Under these conditions one can construct only approximate locally optimal plans using *a priori* information on the values of the parameters p_k, $k = \overline{1, M}$ [3, 4].

The boundary problems of Eqs. (13)-(16) are linear. To solve them one must know the temperature field $T(x, \tau)$, and therefore these problems are solved simultaneously with the original problem of Eqs. (1)-(4) with given estimates of the unknown parameters. These boundary problems are solved numerically using a monotonic finite-difference approximation scheme [7]. Here for all the problems we used the same space–time difference mesh, chosen to secure the required accuracy of numerical solution of the corresponding IBP.

To seek the optimal thermal sensor coordinate vector **X** for a fixed number of sensors N from condition (12) we use a scanning method [8] in the given spatial mesh. The search procedure consists of successive calculation and comparison of the values of the criterion of Eq. (9) at the mesh nodes. This results in determining the global extremum of the criterion and the optimal vector **X** is calculated to an accuracy of half a mesh step [5, 6].

In carrying out a real experiment the actual coordinates of the spatial location of sensors can differ from the optimal values, e.g., because of the mounting inaccuracy. Therefore, along with the choice of optimal measurement plans, one should analyze the sensitivity of the criterion (9) to possible variations of the coordinates X_i, $i = \overline{1, N}$. This analysis is based on investigating the dependence of the optimal criterion on the location of the i-th sensor alone $\Psi(X_i)$, $i=\overline{1, N}$, for fixed coordinates of the remaining sensors. Analysis of the sensitivity should be considered as an inseparable part of the optimal planning problem.

Using the algorithm described we developed a computer program with which we numerically solved a number of temperature measurement optimization problems. We examined problems of choosing mounting coordinates for a varying number of thermal sensors and analyzed the sensitivity in recovering heat flux density at the left boundary of the plate $q_1(\tau)$, and also with simultaneous determination of the heat flux densities on the two plate boundaries $q_1(\tau)$ and $q_2(\tau)$. As basic functions we used cubic B-splines [9].

In the first example the original data were [10]: $b=1$, $\tau_m=1$, $\lambda(T)=1$, $C(T)=1$, $T_0(x)=0$. In the calculations we assumed *a priori*: $q_1(\tau)=1$, $q_2(\tau)=0$. Here it was postulated that there were no measurement errors. The temperature field and the sensitivity functions were calculated on a uniform space—time mesh with number of nodes $n_x \times n_\tau = 21 \times 21$.

Figure 1 shows the dimensionless criterion $\overline{\Psi}(\mathbf{X}) = \Psi/\Psi_{max}$ in recovering the boundary function $q_1(\tau)$ as a function of the measured temperature data at N = 1, 2, and 3 points, respectively, and number of parameters $m_1 = 4$. Figure 2 shows the dependence $\overline{\Psi}(\mathbf{X})$ for the case of simultaneous determination of $q_1(\tau)$ and $q_2(\tau)$ with N = 2, 3 and M = 8 ($m_1 = 4$ and $m_2 = 4$).

We also compared the measurement planning results with mathematical modeling data in which the boundary problem was solved with different measurement point coordinates [2].

Figure 3 shows the recovered dependence $q_1(\tau)$ in the dimensionless coordinates $\overline{q_1} = q_1/q_{1max}$ ($q_{max} = 10.5$ kW/m²) and $\bar{\tau} = \tau/\tau_m$. The planning problem was solved with the following original data: b = 0.1 m, $\tau_m = 1700$ sec, $a = 1.09 \cdot 10^{-6}$ m²/sec, $T_0(x) = 300$ K, $q_2(\tau) = 0$, $n_x \times n_\tau = 41 \times 41$ and M = 6. The results are shown in Fig. 4. Here the relative integral error in recovering the heat flux density was calculated from the formula [2]

$$\rho_q = \left[\int_0^{\tau_m} [q_1(\tau) - \tilde{q}_1(\tau)]^2 d\tau \Big/ \int_0^{\tau_m} q_1^2(\tau)\, d\tau\right]^{1/2},$$

where $q_1(\tau)$ is the exact dependence; and $\tilde{q}_1(\tau)$ is the recovered function.

The results obtained show that it is optimal to locate the thermal sensors on the surface on which a specific heat flux acts. This sensor location leads to the case of a pseudo-inverse problem [1]. As the measurement points become increasingly distant from the heated body surface the system sensitivity decreases, leading to a degraded condition for the inverse problem being analyzed, and, as a result, to degradation of the numerical properties (e.g., rate of convergence of the iterative process). By analyzing the sensitivity one can select regions for preferred thermal sensor location, and also regions where the system sensitivity tends to zero and the experimental information is not enough for the IBP to be solved with the required accuracy.

The use of additional measurements increases the system sensitivity and improves the computational properties of the problem. However, as one increases the number of thermal sensors the influence of N on the system sensitivity decreases.

The measurement planning results agree well with the results of numerical solution of the IBP.

NOTATION

T, temperature; x, coordinate; τ, time; τ_m, process duration; $T_0(x)$, initial temperature distribution; C(t), volume specific heat; $\lambda(T)$, thermal conductivity; a, thermal diffusivity; b, plate thickness; $q_1(\tau)$, $q_2(\tau)$, heat flux densities on the left and right boundaries of the plate, respectively; N, number of thermal sensors; $f_i(\tau)$, $i = \overline{1, N}$, experimentally measured temperatures; $\mathbf{X} = \{X_i, i = \overline{1, N}\}$, vector of the thermal sensor mounting coordinates; $\varepsilon = \{N, \mathbf{X}\}$, plan of the measurements; φ_k, $k = \overline{1, M}$, system of basic functions; p_k, $k = \overline{1, M}$, coefficients of the approximate relation; I, functional; δ^2, generalized error of the temperature measurements; Ψ, optimal criterion; $F(\varepsilon)$, normalized Fisher information matrix; $\Theta_k(x, \tau)$, $k = \overline{1, M}$, sensitivity function; ρ_q, relative integral error of recovering the heat flux density. Subscripts: max, min, maximum and minimum values, respectively.

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DETERMINATION OF THE THERMOPHYSICAL PROPERTIES OF TRANSLUCENT MATERIALS

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A method is proposed for determining the thermophysical properties of translucent scattering materials in the nonsteady heating regime.

Translucent materials capable of selectively reflecting, transmitting, absorbing, and scattering radiation from external heat sources and background ratiation are in use in a number of thermally loaded structures and are being considered for more such applications. The empirical literature data on the thermophysical properties (TPP) of translucent scatterers in the high-temperature region — where radiative heat transfer is important — is of an approximate nature. Heat transfer occurs in translucent materials simultaneously by conduction and radiation, and the temperature and radiation fields in the materials are coupled. Thus, without isolation of the individual components of heat transfer — conductive and radiative — experimental data on the thermal conductivity and diffusivity of translucent materials cannot be widely used in heat-engineering calculations because they apply only to specific empirical conditions of heat transfer for the given specimen.

The feasibility of using well-known experimental methods of the thermophysics of the optical media [1] to correctly determine TPP and to isolate the individual components of heat transfer in translucent scattering materials is problematic for several reasons. First, mathematical models of inverse coefficient problems of radiative—conductive heat transfer (IPRCT) do not consider such important features of heat transfer as multiple scattering of radiation in absorbing and radiating media. Second, translucent scatterers are generally poor heat conductors. For these materials, as for other thermal insulators, despite the volumetric character of heating it is possible to create small temperature gradients and heating rates in the specimen only in a long experiment employing complicated equipment. We add that determining the optical properties of translucent scatterers at high temperatures is a complicated problem by itself. In this connection, it is important to develop new experimental methods that will make it possible to efficiently determine the TPP of translucent scattering materials in the regime of intensive nonsteady heating.

We will examine a physical and mathematical model of heat transfer in a translucent scattering material for the conditions of stand heat-engineering tests [2].

We will assume that the frontal surface of the plane specimen of isotropic translucent scattering material is heated by a radiation flow of a known spectral composition and density. The coefficient of heat transfer to the gaseous medium on the front surface and the temperature dependence of the optical properties of the material (absorption coefficient α, scattering coefficient β, and refractive index n) are assumed to be known. The rear surface of the specimen is thermally insulated. Experimental thermograms are taken at one or several points of the specimen during heating. It is necessary to determine the temperature dependence of the thermal conductivity and volumetric specific heat of the material.

Heat transfer in a translucent scattering material is described by a system of equations which includes the equations of heat conduction and radiation and the corresponding boundary

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